

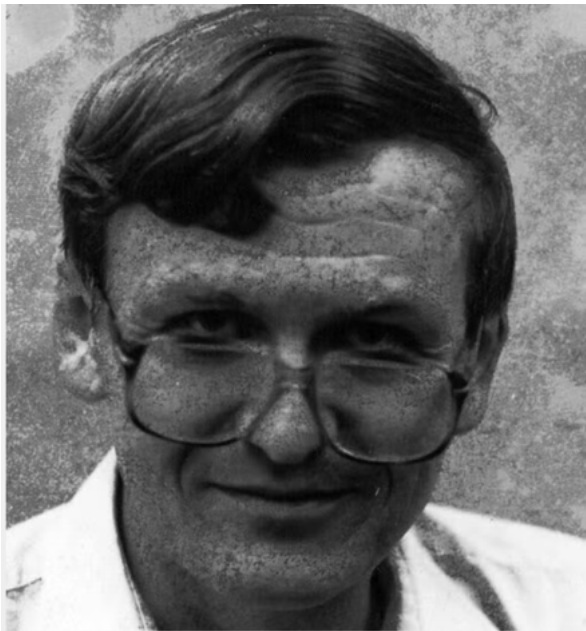
From p -groups to defect groups to nilpotent blocks

Gabriel Navarro

Shenzhen, SUSTech, Broué's Conference, May 2026



VNIVERSITAT
DE VALÈNCIA



Everett C. Dade



Local methods in block theory

By J. ALPERIN¹ and MICHEL BROUÉ²

1. Introduction

In a series of fundamental papers [2]-[8], Brauer made many great discoveries about the structure of blocks of characters. In all of this work the local subgroups played a prominent role. At the same time basic discoveries have been made in local group theory. In this paper we shall give a synthesis of these two developments. In particular, one of our results is a common generalization of Brauer's First and Third Main Theorems and of the Sylow theorems. Another generalizes Brauer's results on the conjugacy of subsections and the theory of local fusion.

Inventiones math. 56, 117 – 128 (1980)

*Inventiones
mathematicae*

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A Frobenius Theorem for Blocks

Michel Broué¹ and Lluís Puig²

¹ 9 rue Brézin, F-75014 Paris, France

² 3 rue Jules-Guesde, F-94140 Alfortville, France

Théorie des groupes/*Group Theory*

Blocs, isométries parfaites, catégories dérivées

Michel BROUÉ

Résumé — Une équivalence de catégories dérivées d'algèbres de groupes finis sur un anneau l -adique produit une isométrie possédant des propriétés arithmétiques intéressantes entre les groupes de caractères. On étudie ces isométries, et on donne des exemples d'applications.

Blocks, perfect isometries, derived categories

Abstract — An equivalence of the derived categories of finite group algebras over an l -adic ring provides an isometry between character groups with nice arithmetical properties. We study such isometries, and we give some applications.



October, 2006.

Background on Brauer p -blocks

Let p be a prime. We choose an appropriate local ring $\mathcal{O}$: let M be maximal ideal of the ring of algebraic integers $R \subset \mathbb{C}$ with $p \in M$, and let $\mathcal{O}$ be the localization of R at M :

$$\mathcal{O} = \{ \alpha / \beta \mid \alpha \in R, \beta \in R - M \} .$$

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$$\mathcal{O} = \{ \alpha / \beta \mid \alpha \in R, \beta \in R - M \}.$$

Then $\mathcal{O}$ has a unique maximal ideal

$$\mathcal{J} = \{ \alpha / \beta \mid \alpha \in M, \beta \in R - M \}$$

and $\mathcal{O} / \mathcal{J} = F$ is the algebraic closure of the field with p elements.

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$$* : \mathrm{GL}_n(\mathcal{O}) \longrightarrow \mathrm{GL}_n(F),$$

$$* : \mathcal{O}G \longrightarrow FG,$$

where $\mathcal{O}G$ is the free $\mathcal{O}$ -algebra with basis G , where G is a finite group.

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If G is a finite group, we decompose

$$\mathcal{O}G = B_1 \oplus \cdots \oplus B_t,$$

where the B_i are indecomposable two-sided ideals. The B_i 's are the **p -blocks** of G .

If

$$1 = f_1 + \cdots + f_t,$$

where $f_i \in B_i$, then B_i is an $\mathcal{O}$ -algebra with identity f_i .

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We also have that

$$FG = (B_1)^* \oplus \cdots \oplus (B_t)^*,$$

and

$$1 = e_1 + \cdots + e_t,$$

where $e_i = (f_i)^*$ is called the **block idempotent** of B_i .

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The principal character lies in the **principal block**, the most important block.

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For the principal block, $D \in \text{Syl}_p(G)$.

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If P is any p -subgroup of G , $C = C_G(P)$, then it is easy to prove that the map

$$\mathrm{Br}_P : Z(FG) \rightarrow Z(FC)$$

which sends $\hat{K} \mapsto \widehat{K \cap C}$ defines an algebra homomorphism, where $K \in \mathrm{Cl}(G)$, and $\widehat{K \cap C} := 0$ if $K \cap C$ is empty. (If $X \subseteq G$, $\hat{X} = \sum_{x \in X} x$.)

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where $|G|_p = p^a$, $|D| = p^d$, and $h = h(\chi) \geq 0$ is the **height** of χ . In the most important case where $D \in \text{Syl}_p(G)$, $\chi(1)_p = p^{h(\chi)}$.

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- The p -blocks are the connected components.

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There is a canonical bijection

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Some data for $p = 2$ and principal blocks $B = b^G$ of G , b of $\text{N}_G(P)$, $P \in \text{Syl}_p(G)$, where $k(B) = |\text{Irr}(B)|$:

A5 $k(B)=4$ $k(b)=4$
PSL(3,2) $k(B)=5$ $k(b)=5$
A6 $k(B)=5$ $k(b)=5$
A7 $k(B)=5$ $k(b)=5$
PSL(3,3) $k(B)=8$ $k(b)=7$
PSU(3,3) $k(B)=12$ $k(b)=14$
M11 $k(B)=8$ $k(b)=7$
A8 $k(B)=13$ $k(b)=16$
PSL(3,4) $k(B)=9$ $k(b)=9$
O(5,3) $k(B)=19$ $k(b)=16$
Sz(8) $k(B)=10$ $k(b)=10$
PSU(3,4) $k(B)=21$ $k(b)=21$
M12 $k(B)=11$ $k(b)=16$

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Alperin Weight conjecture, the p odd case of Alperin–McKay, Dade's conjectures and Broué's conjecture (and a possible generalization) are the **main** problems in global/local block theory, right now.

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If χ_i belongs to a p -block B of defect d , it follows from the definition of defect, that the integer $h_i = d - \nu_p(n/\chi_i(1))$ is non-negative. We call h_i the height of χ_i (for given p). Our next question then is:

Problem 23. Is it true that a block B has Abelian defect group D , if and only if all $\chi_i \in B$ have height 0?

Birds and Frogs

Freeman Dyson

Some mathematicians are birds, others are frogs. Birds fly high in the air and survey broad vistas of mathematics out to the far horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects, and they solve problems one at a time. I happen to be a frog, but many of my best friends are birds. The main theme

From p -groups to defect groups: Heights

If D is a p -group and $\chi \in \text{Irr}(D)$,
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COUNTEREXAMPLE

$G = GL_2(3)$, $p = 2$, $B = B_0(G)$, $\chi \in \text{Irr}(B)$ has height 2, and $D = QD_{16}$ has a cyclic normal subgroup A of index 2.

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Is there something between being central and being normal and abelian?

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Suppose that B is a p -block with defect group D . Write $|D| = p^d$.

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For G solvable, $p = 2$, unknown.

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QUESTION

If $\chi \in \text{Irr}(B)$ has height h , is it $2h \leq d$?

THEOREM (Moretó-N)

If G is p -solvable, p odd, then $2h \leq d$.

COUNTEREXAMPLE

$G = 2.A_8$, $p = 2$, $B_0(G)$,
 $|G|_p = 2^7$, $h(\chi) = 4$. Or Co_1 ,
 $p = 3$; M , $p = 5$.

For G solvable, $p = 2$, unknown.

$G = \text{SL}_n(p)$ has characters with p -height $h = (n-1)(n-2)/2$, while $d = n(n-1)/2$ so $h/d \rightarrow 1$.

From p -groups to defect groups: Heights

If D is a non-abelian p -group, let $mh(D)$ be the smallest positive height of D (that is, the smallest non-zero exponent of degrees), and let $mh(B)$ be the smallest positive height in $\text{Irr}(B)$, a p -block with defect group D .

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- (N, 2025) True if G is p -solvable and B is the principal block.

QUESTION

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This is Eaton–Moretó in the important case $F^*(G) = F(G)$, for principal p -blocks. I have not been able to find extra-relations between the $\tau \in \text{Irr}(D)$ with minimal degree > 1 and the $\chi \in \text{Irr}(B|D)$ with minimal height > 0 . (Number of, field of values of, etc.)

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In these groups the minimal degree of the Sylow 2-subgroup is not 2 but 4...

From p -groups to defect groups: Heights

Learning from p -groups:

If a_n is the number of non-abelian groups of order p^n and b_n is the number of p -groups of order p^n having an irreducible character of degree p :

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And an analog with heights?

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Let $\text{cd}(X)$ be the set of character degrees of the irreducible characters of the finite group X , or of the block X .

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COUNTEREXAMPLE

$Q = \text{SmallGroup}(3^6, 122)$, $A = C_{13} : 3$ acts on Q . Let $G = QA$. Then $\text{cd}(G) = \{1, 3, 13, 39\}$, $p = 3$, $B = B_0(G)$ (so heights $\{0, 1\}$) while $\text{dl}(D) = 3$.

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- (Giannelli-Martinez-Schaeffer Fry) If $|\text{cd}(B_0(G))| \leq 3$, then $\text{dl}(D) \leq |\text{cd}(B_0(G))|$.

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In general, $k(B) \not\leq k(D)$: $G = \text{SL}_2(3)$, $p = 2$, $B = B_0(G)$.

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This is the $F^*(G) = F(G)$ case of Brauer's $k(B)$ for the principal block, the next $k(GV)$ -theorem. Even if this is solved, what about the case where $F^*(G)$ is the product of non-abelian simple groups? $k(b^G)$ can be smaller or bigger than $k(b)$.

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for all $s \geq 0$.

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It is not true necessarily that $h_t(B) \leq h_t(D)p^{2t}$. For instance, $GL_2(3)$, $p = 2$, $h_2(B_0(G)) = 1$ and $h_2(D) = 0$.

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QUESTION

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If $p = 2$, $B = B_0(G)$,

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QUESTION (Malle-N)

$$\text{Is } k(B) \leq k(D')h_0(B)?$$

Assuming Olsson $h_0(B) \leq |D/D'|$, the weaker inequality $k(B) \leq |D'|h_0(B)$ already implies the $k(B)$ -conjecture.

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QUESTION

If $x \in D$, is there some p -rational $\chi \in \text{Irr}(B)$ such that $\chi(x) \neq 0$?

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We say that $\chi \in \text{Irr}(G)$ is **p -rational** if $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_n = \mathbb{Q}(e^{2\pi i/n})$, where p does not divide n .

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According to GAP (and T. Breuer), 2.SuzM7 is a counterexample for $p = 3$ (the 15th block).

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CONJECTURE

If $x \in D$, then there is $\chi \in \text{Irr}(B)$ of height zero almost- p -rational such that $\chi(x) \neq 0$.

From p -groups to defect groups: $Z(D)$

QUESTION

If we know the character table of G , and B is a p -block of G , can we detect the p -elements in $Z(D)$?

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QUESTION

If B is a block with defect group D and $x \in D$, does x lie in some G -conjugate of $Z(D)$ iff $\chi(x) \neq 0$ for all $\chi \in \text{Irr}(B)$?

Suppose that $x \in Z(D)$. Do we have that $\chi(x) \neq 0$ for all $\chi \in \text{Irr}(B)$? (Brauer, Geoff Robinson?:)

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$$\left(\frac{|G : C_G(x)|\chi(x)}{\chi(1)} \right)^* = |G : C_G(x)|^* \neq 0.$$

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Not even in the cyclic defect case: $SL_2(11)$ has a 2-block B with $D = C_4 = \langle x \rangle$, and there is $\chi \in \text{Irr}(B)$ with $\chi(x) = 0$.

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Suppose that $x \in Z(D)$ and $\chi \in \text{Irr}(B)$. Then χ is not zero on the p -section of x .

What about the converse?

```
CharacterTable( "Sz(8)" )
```

```
gap> Display(x);
```

```
Sz(8)
```

2	6	6	4	4	.	.	.	.	.	.	.
5	1	.	.	.	1	.	.	.	.	.	.
7	1	.	.	.	.	1	1	1	.	.	.
13	1	.	.	.	.	.	.	.	1	1	1

	1a	2a	4a	4b	5a	7a	7b	7c	13a	13b	13c
2P	1a	1a	2a	2a	5a	7b	7c	7a	13b	13c	13a
5P	1a	2a	4a	4b	1a	7b	7c	7a	13a	13b	13c
7P	1a	2a	4b	4a	5a	1a	1a	1a	13c	13a	13b
13P	1a	2a	4a	4b	5a	7a	7b	7c	1a	1a	1a

X.1	1	1	1	1	1	1	1	1	1	1	1
X.2	14	-2	A	-A	-1	.	.	.	1	1	1
X.3	14	-2	-A	A	-1	.	.	.	1	1	1
X.4	35	3	-1	-1	.	.	.	.	E	G	F
X.5	35	3	-1	-1	.	.	.	.	F	E	G
X.6	35	3	-1	-1	.	.	.	.	G	F	E
X.7	64	.	.	.	-1	1	1	1	-1	-1	-1
X.8	65	1	1	1	.	B	D	C	.	.	.
X.9	65	1	1	1	.	C	B	D	.	.	.
X.10	65	1	1	1	.	D	C	B	.	.	.
X.11	91	-5	-1	-1	1	.	.	.	.	.	.

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7P	1a	2a	4b	4a	5a	1a	1a	1a	13c	13a	13b
13P	1a	2a	4a	4b	5a	7a	7b	7c	1a	1a	1a

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$$Z(D) = C_2 \times C_2 \times C_2.$$

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$$Z(D) = C_2 \times C_2 \times C_2.$$

So the answer is that I do not know how to detect that $x \in Z(D)$.

From p -groups to defect groups: Exponents

Suppose that B is a p -block with defect group D . Write $|D| = p^d$.

THEOREM

If $\chi \in \text{Irr}(D)$ and $x \in D$ then $o(x)$ divides $|D|/\chi(1)$.

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If $\chi \in \text{Irr}(B|D)$, is $\exp(D) \leq |G|_p/\chi(1)_p$?

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If $\chi \in \text{Irr}(B|D)$, then $\exp(Z(D)) \leq |G|_p/\chi(1)_p$.

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We say that a block B with defect group D satisfies the $\dagger$ condition if:

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Why do we care about this?

A beautiful conjecture:

CONJECTURE (Wilde)

Let $\chi \in \text{Irr}(G)$. If $\chi(g) \neq 0$, then $o(g)$ divides $|G|/\chi(1)$.

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If the block of χ satisfies condition $\dagger$ (for all p), then the conjecture is true for G . (Recall: $\dagger$ is that $\exp(D) \leq |G|_p/\chi(1)_p$ for every $\chi \in \text{Irr}(B)$.)

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THEOREM (Malle-N-Tiep, 2026)

Condition $\dagger$ is satisfied by many blocks of quasi-simple groups, but not always.

THEOREM (Malle-N-Tiep, 2026)

Suppose that whenever $h \in H$, $[H, H] = L$ is quasisimple, $Z(L) = Z(H)$ is cyclic, $H = L\langle h \rangle$, and $\chi \in \text{Irr}(H)$ is faithful with $\chi(h) \neq 0$, then $o(hZ(H))$ divides $|H : Z(H)|/\chi(1)$. Then Wilde's conjecture is true.

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QUESTION

If all $\text{Irr}(\Phi_1)$ have p' -degree, P is metabelian?

Nilpotent blocks

No blocks are closer to their defect groups than the so-called nilpotent blocks. They are ubiquitous in block theory.

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DEFINITION (Broué-Puig)

*A p -block B is **nilpotent** if for every p -subgroup Q and every p -block b_Q of $QC_G(Q)$ inducing B , we have that $N_G(Q, b_Q)/C_G(Q)$ is a p -group.*

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If $B = B_0(G)$ is the principal block of G , B is nilpotent if and only if $N_G(Q)/C_G(Q)$ is a p -group for all p -subgroup Q of G . Hence, by Frobenius Theorem:

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THEOREM

$B_0(G)$ is nilpotent if and only if G has a normal p -complement.

From defect groups to nilpotent blocks

A local definition that gives rise to a wonderful theorem:

THEOREM (Broué–Puig)

Suppose that $B \in \text{Bl}(G|D)$ is nilpotent. Then there exists a p -rational $\varphi \in \text{Irr}(B)$ height zero, such that there is a bijection $\eta \mapsto \eta * \varphi$ from $\text{Irr}(D) \rightarrow \text{Irr}(B)$.

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In the $*$ -construction, we have that $(\eta * \varphi)(1) = \eta(1)\varphi(1)$. Hence the heights in $\text{Irr}(B)$ are exactly the heights in $\text{Irr}(D)$, in bijection.

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In particular, $\text{dl}(D) \leq |\text{cd}(B)|$, if B is nilpotent.

If D is abelian and B is nilpotent, then $\text{cd}(B) = \{n\}$, so these are Okuyama–Tsushima blocks.

From defect groups to nilpotent blocks

With a lot of effort:

From defect groups to nilpotent blocks

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J. reine angew. Math. **656** (2011), 131—177
DOI 10.1515/CRELLE.2011.046

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Nilpotent blocks of quasisimple groups for odd primes

By *Jianbei An* at Auckland and *Charles W. Eaton* at Manchester

Abstract. We investigate the nilpotent blocks of positive defect of the quasisimple groups for odd primes. In particular, it is shown that every nilpotent block of a quasisimple group has abelian defect groups. A conjecture of Puig concerning the recognition of nilpotent blocks is also shown for these groups.

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Also for $p = 2$.

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Invent. math. 102, 17–71 (1990)

*Inventiones
mathematicae*
© Springer-Verlag 1990

Extensions of nilpotent blocks[★]

Burkhard Külshammer¹ and Lluís Puig²

¹ Universität Dortmund, Fachbereich Mathematik, Postfach 500 500, D-4600 Dortmund 50, Federal Republic of Germany

² CNRS, Université de Paris 7, DMI Ecole Normale Supérieure, 45 rue d'Ulm, F-75005 Paris, France

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This was key in the proof of the Height Zero Conjecture.

From defect groups to nilpotent blocks

But how do we recognize nilpotent blocks?

From defect groups to nilpotent blocks

But how do we recognize nilpotent blocks?

```
gap> x:=CharacterTable("M23");
CharacterTable( "M23" )
gap> Display(x);
M23
```

2	7	7	2	5	.	2	1	1	3	.	.	1	1	.	.	.	.
3	2	1	2	.	1	1	.	.	.	.	.	.	1	1	.	.	.
5	1	.	1	.	1	.	.	.	.	.	.	.	1	1	.	.	.
7	1	1	.	.	.	1	1	.	.	.	.	1	1	.	.	.	.
11	1	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.
23	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.

	1a	2a	3a	4a	5a	6a	7a	7b	8a	11a	11b	14a	14b	15a	15b	23a	23b
2P	1a	1a	3a	2a	5a	3a	7a	7b	4a	11b	11a	7a	7b	15a	15b	23a	23b
3P	1a	2a	1a	4a	5a	2a	7b	7a	8a	11a	11b	14b	14a	5a	5a	23a	23b
5P	1a	2a	3a	4a	1a	6a	7b	7a	8a	11a	11b	14b	14a	3a	3a	23b	23a
7P	1a	2a	3a	4a	5a	6a	1a	1a	8a	11b	11a	2a	2a	15b	15a	23b	23a
11P	1a	2a	3a	4a	5a	6a	7a	7b	8a	1a	1a	14a	14b	15b	15a	23b	23a
23P	1a	2a	3a	4a	5a	6a	7a	7b	8a	11a	11b	14a	14b	15a	15b	1a	1a

X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	22	6	4	2	2	.	1	1	.	.	-1	-1	-1	-1	-1	-1	-1
X.3	45	-3	.	1	.	.	A	/A	-1	1	1	-A	-/A	.	.	-1	-1
X.4	45	-3	.	1	.	.	/A	A	-1	1	1	-/A	-A	.	.	-1	-1
X.5	230	22	5	2	.	1	-1	-1	.	-1	-1	1	1	.	.	.	.
X.6	231	7	6	-1	1	-2	.	.	-1	.	.	.	.	1	1	1	1
X.7	231	7	-3	-1	1	1	.	.	-1	.	.	.	.	C	/C	1	1
X.8	231	7	-3	-1	1	1	.	.	-1	.	.	.	.	/C	C	1	1
X.9	253	13	1	1	-2	1	1	1	-1	.	.	-1	-1	1	1	.	.
X.10	770	-14	5	-2	.	1	.	.	.	.	.	.	.	.	.	D	/D
X.11	770	-14	5	-2	.	1	.	.	.	.	.	.	.	.	.	/D	D
X.12	896	.	-4	.	1	.	.	.	.	B	/B	.	.	1	1	-1	-1
X.13	896	.	-4	.	1	.	.	.	.	/B	B	.	.	1	1	-1	-1
X.14	990	-18	.	2	.	.	A	/A	.	.	.	A	/A	.	.	1	1
X.15	990	-18	.	2	.	.	/A	A	.	.	.	/A	A	.	.	1	1
X.16	1035	27	.	-1	.	.	-1	-1	1	1	1	-1	-1	.	.	.	.
X.17	2024	8	-1	.	-1	-1	1	1	.	.	.	1	1	-1	-1	.	.

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Suppose now that $\text{cd}(B) = \{n, m\}$. Suppose that $n_p \leq m_p$. If $n_p = m_p$, then D is abelian by the Height Zero conjecture. If $n_p < m_p$, then all height zero characters have degree n , and therefore B is (conjecturally by Malle-N) nilpotent. Thus $|\text{cd}(D)| = 2$, and D is metabelian.

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Wide open. Do not mess up with decomposition numbers!

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The first part would be a consequence of $k(B) \leq k(D)l(B)$ for every block, suggested by Malle-N.

Representations of Finite Groups

Richard Brauer

INTRODUCTION

It has been said by E. T. Bell that “wherever groups disclosed themselves or could be introduced, simplicity crystallized out of comparative chaos.” This may often be true, but, strangely enough, it does not apply to group theory itself, not even when we restrict ourselves to groups of finite order. We are reminded of the educators who want to educate the world and cannot handle their own children. A tremendous effort has been made by mathematicians for more than a century to clear up the chaos in group theory. Still, we cannot answer some of the simplest questions.